Minimum safe distance

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This document calculates the minimum safe distance when two cars drive behind each other and the first car suddenly brakes. The work presented here is based on the original work *On a Formal Model of Safe and Scalable Self-driving Cars* by Shashua et. al [1]. In fact here we only correct for a small error in the original derivation of the minimum safe distance, which was discovered and published on LinkedIn by Thusitha Parakrama from *Mobis Parts Europe*. Finally we show some more details of the derivation itself, hoping to improve the readability for those readers whose elementary physics classes have been quite a while ago.

1 Basic physics

We start with the kinematics of a point mass, which we'll later simply expand with the size of a car. First basic assumption: we deal with *constant* accelerations, regardless of acceleration or deceleration[1]. Starting with a constant acceleration $a(t) = a_0 = \text{const.}$ we have

$$a(t) = a_0 = \text{const.} \tag{1}$$

$$v(t) = \int a(t)\mathrm{d}t = a_0 t + v_0 \tag{2}$$

$$x(t) = \int v(t)\mathrm{d}t = \frac{1}{2}a_0 t^2 + v_0 t + x_0, \tag{3}$$

where $a(t)$, $v(t)$ and $x(t)$ are acceleration, velocity and position of a car, and x_0 and v_0 are the starting distance and velocity, respectively, at $t = 0$. Therefore when calculating the distance for any car we always have three contributors:

1. some starting distance x_0, that often will be taken together with other constants,
2. the distance due to the velocity at the beginning (at *that* time $\Rightarrow$ which is the source of the error)
3. the distance due to the acceleration.

Eq. 3 is the important one: the whole derivation of the safe minimum distance is based on it, and we will always find these three contributors.

[1] From now on acceleration is treated in the physical sense: it means both acceleration and deceleration, the direction given by the sign of the value. This will be important for the signs later on.

2 Derivation

We use the notation of the original paper, only slightly adjusted. Indices r and f always indicate the rear or the front vechicle, respectively. One important difference is that we include the sign of the acceleration within the actual physical quantity. In the Mobileye paper the numerical value of the physical quantity *acceleration* is always positive, and the sign is taken into account in the formulae. Therefore beware: we always write + acceleration, regardless of acceleration or deceleration, and Mobileye writes + for acceleration and − for deceleration.

The relevant terms are:

- $c_\mathrm{r}, c_\mathrm{f}$: the cars themselves, to have a name
- $a_\mathrm{a}, a_\mathrm{b}$: acceleration (index a) and deceleration (index b, braking)
- $v_\mathrm{r}(t), v_\mathrm{f}(t)$: velocities for a given time t.
- $v_{\mathrm{r},0}, v_{\mathrm{f},0}$: starting velocities of the cars
- $v_{\mathrm{r,\ max}}$: the maximum velocity reached by the rear vehicle
- $x_\mathrm{r}(t), x_\mathrm{f}(t)$: position of the cars for a given time t.
- $x_{\mathrm{r},0}, x_{\mathrm{f},0}$: starting velocities of the cars
- $t_\mathrm{f}, t_\mathrm{r}, t_\mathrm{react}$: the time until c_f and c_r come to stand still, and the reaction time.

Note that we assume that both cars have the same maximum acceleration and deceleration. Taking this into account in the calculation would be straight-forward but reduce readability.

Another assumption is that the rear car takes longer to stop, which reflects reality because of the reaction time, therefore $t_\mathrm{r} > t_\mathrm{f}$. In the Mobileye paper the whole process is then divided into three time segments:

1. During the reaction time: $0 \le t \le t_\mathrm{react}$,
2. after reaction time until the front car stops: $t_\mathrm{react} < t \le t_\mathrm{f}$,
3. and finally until the rear car stops: $t_\mathrm{f} < t \le t_\mathrm{r}$.

To ensure that the safe distance is a worst case assumption the rear car is actually maximally accelerating during the reaction time. Therefore the cars do the following in the three segments:

1. c_f is maximally braking, c_r is maximally accelerating.
2. c_f continues to maximally brake until stand still, c_r is now maximally braking.
3. c_f is at stand still, c_r is maximally braking until stand still.

Now the position of the front car is very easy to determine, simply take Eq. 3 and insert the appropriate symbols. It brakes until t_f is reached and then stands still, therefore

$$x_\mathrm{f}(t_\mathrm{f}) = \frac{1}{2} a_\mathrm{b} t_\mathrm{f}^2 + v_{\mathrm{f},0} t_\mathrm{f} + x_{\mathrm{f},0}. \tag{4}$$

The position of the rear car has two components. After reaction time it is

$$x_{\mathrm{r}}(t_{\mathrm{react}}) = \frac{1}{2}a_{\mathrm{a}}t_{\mathrm{react}}^2 + v_{\mathrm{r},0}t_{\mathrm{react}} + x_{\mathrm{r},0}. \tag{5}$$

After this time the rear car c_{r} has reached its maximum velocity $v_{\mathrm{r,\,max}}$, which is

$$v(t_{\mathrm{react}}) = v_{\mathrm{r,\,max}} = v_{\mathrm{r},0} + a_{\mathrm{a}}t_{\mathrm{react}} \tag{6}$$

Therefore after deceleration the position of c_{r} is

$$x_{\mathrm{r}}(t_{\mathrm{r}}) = \frac{1}{2}a_{\mathrm{b}}(t_{\mathrm{r}} - t_{\mathrm{react}})^2 + v_{\mathrm{r,\,max}}(t_{\mathrm{r}} - t_{\mathrm{react}}) + x_{\mathrm{r}}(t_{\mathrm{react}}). \tag{7}$$

And here is the point where the original paper makes an error: $x_{\mathrm{r}}(t_{\mathrm{react}})$ is not taken into account correctly. It is best seen in the proof to Lemma 2 on page 6. The authors write in the first line for d_T (in their original notation)

$$\rho\, v_{\rho,\mathrm{max}} \tag{8}$$

for the distance already travelled by c_{r} during reaction time. In the notation used here this is $t_{\mathrm{react}} \cdot v_{\mathrm{r,\,max}}$. In other words, they assume that the car was traveling at maximum velocity during all of the reaction time, ignoring the acceleration during that phase. Going back to our notation again let's look at the difference this makes:

$$\begin{aligned} t_{\mathrm{react}}v_{\mathrm{r,\,max}} &= t_{\mathrm{react}}(v_{\mathrm{r},0} + a_{\mathrm{a}}t_{\mathrm{react}}) && (9)\\ &= a_{\mathrm{a}}t_{\mathrm{react}}^2 + v_{\mathrm{r},0}t_{\mathrm{react}}, && (10) \end{aligned}$$

whereas it should be Eq. 5, repeated here for direct comparison:

$$x_{\mathrm{r}}(t_{\mathrm{react}}) = \frac{1}{2}a_{\mathrm{a}}t_{\mathrm{react}}^2 + v_{\mathrm{r},0}t_{\mathrm{react}} + x_{\mathrm{r},0}. \tag{11}$$

Note that the starting distance in $x_{\mathrm{r},0}$ can be ignored for this discussion about the missing term. Therefore we now have to compare the following:

$$\begin{aligned} \text{Original (Eq. 10):} &\quad a_{\mathrm{a}}t_{\mathrm{react}}^2 + v_{\mathrm{r}}t_{\mathrm{react}} && (12)\\ \text{Here (Eq. 11):} &\quad \frac{1}{2}a_{\mathrm{a}}t_{\mathrm{react}}^2 + v_{\mathrm{r}}t_{\mathrm{react}} && (13) \end{aligned}$$

Apparently in the original paper the distance was too long by $\frac{1}{2}a_{\mathrm{a}}t_{\mathrm{react}}^2$, which is exactly the term that was found by Thusitha Parakrama in his post on LinkedIn.

Summarizing the complete formulae for the position of the both cars are:

$$x_{\mathrm{f}}(t_{\mathrm{r}}) = x_{\mathrm{f}}(t_{\mathrm{f}}) = \frac{1}{2}a_{\mathrm{b}}t_{\mathrm{f}}^2 + v_{\mathrm{f},0}t_{\mathrm{f}} + x_{\mathrm{f},0}. \tag{14}$$

$$\begin{aligned} x_{\mathrm{r}}(t_{\mathrm{r}}) &= \frac{1}{2}a_{\mathrm{b}}(t_{\mathrm{r}} - t_{\mathrm{react}})^2 + v_{\mathrm{r,\,max}}(t_{\mathrm{r}} - t_{\mathrm{react}}) + \frac{1}{2}a_{\mathrm{a}}t_{\mathrm{react}}^2 + v_{\mathrm{r},0}t_{\mathrm{react}} + x_{\mathrm{r},0} && (15)\\ &= \frac{1}{2}(a_{\mathrm{b}} - a_{\mathrm{a}})t_{\mathrm{react}}^2 + (a_{\mathrm{a}} - a_{\mathrm{b}})t_{\mathrm{r}}t_{\mathrm{react}} + \frac{1}{2}a_{\mathrm{b}}t_{\mathrm{r}}^2 + v_{\mathrm{r},0}t_{\mathrm{r}} + x_{\mathrm{r},0} && (16) \end{aligned}$$

with

$$t_{\mathrm{f}} = \frac{v_{\mathrm{f},0}}{a_{\mathrm{b}}}, \tag{17}$$

$$t_{\mathrm{r}} = t_{\mathrm{react}} + \frac{v_{\mathrm{r,\,max}}}{a_{\mathrm{b}}} \tag{18}$$

$$= t_{\mathrm{react}} + \frac{v_{\mathrm{r},0} + t_{\mathrm{react}} a_{\mathrm{a}}}{a_{\mathrm{b}}} \tag{19}$$

From this we readily arrive at the definition of the minimum safe distance (point mass to point mass):

$$d_{\mathrm{min}} = x_{\mathrm{f}}(t_{\mathrm{r}}) - x_{\mathrm{r}}(t_{\mathrm{r}}) \tag{20}$$

$$= \frac{1}{2} a_{\mathrm{b}} t_{\mathrm{f}}^2 + v_{\mathrm{f},0} t_{\mathrm{f}} + x_{\mathrm{f},0} - \left(\frac{1}{2}(a_{\mathrm{b}} - a_{\mathrm{a}}) t_{\mathrm{react}}^2 + (a_{\mathrm{a}} - a_{\mathrm{b}}) t_{\mathrm{r}} t_{\mathrm{react}} + \frac{1}{2} a_{\mathrm{b}} t_{\mathrm{r}}^2 + v_{\mathrm{r},0} t_{\mathrm{r}} + x_{\mathrm{r},0} \right) \tag{21}$$

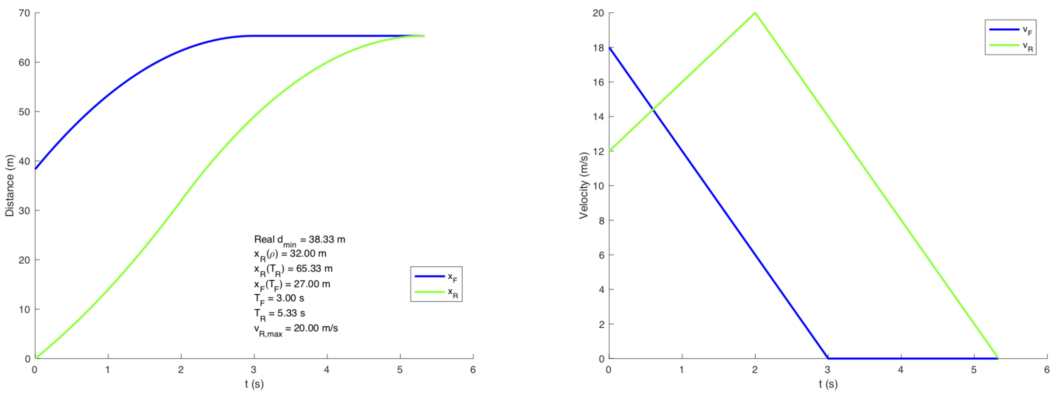
$$= \frac{1}{2} a_{\mathrm{b}} (t_{\mathrm{f}}^2 - t_{\mathrm{r}}^2) + (a_{\mathrm{b}} - a_{\mathrm{a}}) \left(t_{\mathrm{r}} t_{\mathrm{react}} - \frac{1}{2} t_{\mathrm{react}}^2 \right) + (v_{\mathrm{f},0} t_{\mathrm{f}} - v_{\mathrm{r},0} t_{\mathrm{r}}) + (x_{\mathrm{f},0} - x_{\mathrm{f},0}). \tag{22}$$

3 Discussion

Equations 14 —19 give you the exact position of both cars, which are treated as point masses. The absolut minimum safe distance is the difference between the two final positions:

$$d_{\mathrm{min}} = x_{\mathrm{f}}(t_{\mathrm{r}}) - x_{\mathrm{r}}(t_{\mathrm{r}}). \tag{23}$$

Therefore, when the two cars start with a distance d_{min} they will end up right on top of each other, with no distance to spare. We've created a Matlab script that calculates these distances, and sure enough the cars end up on exactly the same spot, as demonstrated in Fig. 1[2].



(a) (b)

Figure 1: (a) Positions $x(t)$ of the two cars, (b) velocities $v(t)$.

Now let's talk about the lengths of the vehicles, and how that should be taken into account when going from point masses to actual voluminous objects. Let us start with a brief remark

[2] The Matlab script can be found at my web page.

about L from the Mobileye paper to clarify notation. $L = (L_\mathrm{r} + L_\mathrm{f})/2$ itself is ill-defined. Imagine c_r to be a truck of length $L_\mathrm{r} = 20\,\mathrm{m}$, and c_f a car of length $L_\mathrm{f} = 4\,\mathrm{m}$. Then the starting distance is per definition $L = 12\,\mathrm{m}$, which – starting from the rear axle – is in the middle of the truck. Clearly an awkward starting point for c_f. But this is of course easily remedied, and not a question of math. Just define the minimum safe distance as the distance between the front bumper of the rear vehicle and the back bumper of the front vehicle, and that's that.

Finally, of course bumper to bumper does seem quite tight. But again this isn't a question of math. The length L is ill-defined, and actually any starting distance based on the lengths of the cars seems to be inappropriate. We propose to use the minimal distance as per exact physics, bumper to bumper, which we've given above, and then add some factor of required minimum tolerance safety. This if of course the same as simply re-defining L, as we said it's about notation. This might be the μ given in the Mobileye paper, or some other to-be-defined distance. This additional factor can then be defined by law-makers, and might be influenced by other requirements (How does that influence the length of a traffic jam? Can people still pass between the standing cars? Should they be able to?).

Let us close by stating that all of the above shows how – even in simple math – you have to take care of every single term. But also note that everything above has nothing to do with *the point* of the Mobileye paper. So the minimum distance wasn't perfectly defined, so what? The whole point is to get away from people driving around in circles for many million miles and then say: *We believe we are safe.* Everyone knows that's good enough for assistance systems, but wholly unsuitable for AD. What is required is a new metric and a new language, so it's possible to show – scientifically and technically sound – that autonomous vehicles are indeed safe. And that is what that paper is about – it's the first prominent proposal how to tackle a serious problem that so far has been neglected, but which needs to be solved to make autonomous driving a success.

References

[1] Shai Shalev-Shwartz, Shaked Shammah, and Amnon Shashua. On a formal model of safe and scalable self-driving cars. *arXiv*, arxiv.org/abs/1708.06374, 2017.