On the connectedness of the Julia-set for rational functions

HANS-GÜNTER MEIER
RWTH Aachen

Lehrstuhl C für Mathematik

Templergraben 55

D-5100 Aachen

Abstract. In this paper the question of the connectedness of the Julia-set for rational functions is considered. A criterion for connectedness motivated by Newton's method for polynomials is proved and discussed in particular for cubic polynomials.

Key words. Julia-set, connectedness, Newton's method, rational functions

AMS (MOS) subject classifications. 30D05

1. Introduction. Since Cayley [5], the question on the behavior of Newton's iteration for finding zeros of polynomials has motivated the investigation of the dynamics of recursive structures.

In particular Julia [11] and Fatou [7] established their theory on the iteration of rational functions, which became of interest again in recent years. Displayed on a computer screen, the complex dynamics may contribute to motivate new theoretical investigations.

In fact, the graphical representation of Newton's method for ploynomials by Curry, Garnett, Sullivan [6], Peitgen, Saupe, v. Haeseler [16] and Benzinger, Burns, Palmore [1] has been the starting point to examine the question whether all components of the appearing Fatou-set are simply-connected or even whether the

Julia-set is connected. This paper tries to analyze the mechanisms which seem to underlie these graphics. After a short recapitulation of elementary results on iterating rational functions, a criterion (Theorem 2) for the connectedness of the Julia-set is developed in section 3. The conditions in this criterion are conjectured to be met by the Newton-iteration for almost all polynomials. In section 4 the latter is discussed for polynomials of degree 3. All results in section 3 are generally applicable to rational functions, and don't need the special structure of Newton's method.

2. Recapitulation of elementary results.

In the following $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$ stands for the Riemann-sphere. A rational function R may be regarded as a mapping from $\mathbb{C}^*$ to $\mathbb{C}^*$ by means of continuous extension. So, a rational function $R: \mathbb{C}^* \to \mathbb{C}^*$ is a holomorphic mapping of the Riemannian surface $\mathbb{C}^*$ into itself [8].

The symbol R^n denotes the n-fold composition $R \circ R \circ \ldots \circ R$ and the forward orbit of a set $A \subseteq \mathbb{C}^*$ related to R is defined by

$$O^+(A) := \{R^n(z_0) \mid n \in \mathbb{N} \wedge z_0 \in A\}.$$

Now, a short summary of definitions and results of the iteration theory for rational functions, which will be used later, is given. Many of these results originate from Julia [11] and Fatou [7].

For a rational function $R: \mathbb{C}^* \to \mathbb{C}^*$ the **Fatou-set** $F(R)$ is defined to be the set of all points $z \in \mathbb{C}^*$ for which there is a neighbourhood U of z where $(R^n{}_{|U})_{n \in \mathbb{N}}$ forms a normal family. We remark that normality refers to compactness. In fact, a family $\mathcal{F}$ of holomorphic functions is called normal if it is relatively compact in a certain function space where convergence is equal to the common local uniform convergence with respect to the chordal metric.

The complement of $F(R)$ is called the **Julia-set** $J(R)$. Let $\alpha \in \mathbb{C}^*$ be a fixed-point of R; notation: $\alpha \in \text{Fix}(R)$. Then, if $\alpha \in \mathbb{C}$, the fixed-point is called

attractive if $|R'(\alpha)| < 1$,

neutral if $|R'(\alpha)| = 1$,

and **repelling** if $|R'(\alpha)| > 1$.

In case $\alpha = \infty$, we are dealing with an attractive (neutral or repelling) fixed-point, if zero is an attractive (neutral or repelling) fixed-point for the continuous extension of $z \mapsto \dfrac{1}{R(\frac{1}{z})}$ to $\mathbb{C}^*$.

If α is an attractive fixed-point, the set

$$A(\alpha) := \{z \in \mathbb{C}^* | \lim_{n\to\infty}(R^n(z)) = \alpha\}$$

is called the **basin of attraction** of α, and the component of $A(\alpha)$ which contains α is called the **immediate basin of attraction** $A^*(\alpha)$ of α.

With these definitions the following Lemma holds. For details we refer to Blanchard [2] and Brolin [4].

LEMMA 1. *Let $R: \mathbb{C}^* \to \mathbb{C}^*$ be a rational function with $\deg(R) \geq 2$ and let $\alpha \in Fix(R)$. Then*

(1) *$J(R) \neq \emptyset$ and $J(R)$ is perfect*

(2) *There is a set $E \subseteq F(R)$, $card(E) \leq 2$, such that for each $z \in J(R)$ and every neighbourhood U of z we have*

$$\mathbb{C}^* \setminus \bigcup_{n=1}^{\infty} R^n(U) \subseteq E$$

(3) *If* α *is attractive, then*

a) $A(\alpha) \subseteq F(R)$ *is open*

b) $J(R) = \partial A(\alpha)$

c) $R(A(\alpha)) = A(\alpha) = R^{-1}(A(\alpha))$

(4) *If* α *is repelling, then* $\alpha \in J(R)$.

(5) $R(J(R)) = J(R) = R^{-1}(J(R))$

(6) *If* α *is attractive, then there is a critical point* c *of* R *(i.e.* $R_{|U}$ *isn't injective in any neighbourhood* U *of* c*) with* $c \in A^*(\alpha)$ *and*

a) $A^*(\alpha)$ *is either simply-connected or of infinite connectivity*

b) $A^*(\alpha)$ *is a component of* $R^{-1}(A^*(\alpha))$.

3. A criterion for the connectedness of the Julia-set.

We start with some fundamental properties of rational functions.

LEMMA 2. *Let* $R: \mathbb{C}^* \to \mathbb{C}^*$ *be a non-constant rational function and let* $\emptyset \neq G \subseteq \mathbb{C}^*$ *be a region.*

Then there is a $k \in \mathbb{N}$ *as well as* $G_1, \dots, G_k \subseteq \mathbb{C}^*$ *which are non-empty and mutually disjoint regions, such that the following holds:*

(1) $R^{-1}(G) = \bigcup_{\nu=1}^{k} G_\nu$ *and* $R(G_\nu) = G$ *for all* $\nu \in \{1, \dots, k\}$.

(2) $R^{-1}(\partial G) = \partial R^{-1}(G) = \bigcup_{\nu=1}^{k} \partial G_\nu$ *and* $R(\partial G_\nu) = \partial G$ *for all* $\nu \in \{1, \dots, k\}$.

(3) *If* G *is simply-connected and if* $R_{|G_\nu}$ *is locally injective for some* $\nu \in \{1, \dots, k\}$, *then* $R_{|G_\nu} : G_\nu \to G$ *is a homeomorphism.*

Remark. The proof of Lemma 2 follows from elementary considerations on proper, holomorphic mappings. Note that the sets $G_1, \dots, G_k$ are precisely the components of the open set $R^{-1}(G)$.

THEOREM 1. *Let* $R:\mathbb{C}^* \to \mathbb{C}^*$ *be a non-constant, rational function, let* $n \in \mathbb{N}$ *and* $G_1, \ldots, G_n \subseteq \mathbb{C}^*$ *non-empty and mutually disjoint regions. For* $m \in \mathbb{N}$ *put*

$$\mathcal{F}_m := \bigcup_{\kappa=1}^{m} \{A \subseteq \mathbb{C}^* | A \text{ is a component of } (R^\kappa)^{-1}(G_\nu) \text{ for some } \nu \in \{1, \ldots, n\}\}$$

and $\mathcal{F} := \bigcup_{m=1}^{\infty} \mathcal{F}_m$. *Suppose that the following conditions hold:*

(1) $G_\nu \in \mathcal{F}_1$ *for each* $\nu \in \{1, \ldots, n\}$

(2) *Each* $A \in \mathcal{F}$ *is simply-connected.*

(3) $\bigcup_{\nu=1}^{n} \partial G_\nu$ *is connected.*

(4) *For each* $A \in \mathcal{F}_1$ *there is a* $\nu_0 \in \{1, \ldots, n\}$, *such that*

$$\partial A \cap \partial G_{\nu_0} \neq \emptyset$$

Then $\bigcup_{A \in \mathcal{F}} \partial A$ *is connected.*

Remark. Note that a region in $\mathbb{C}^*$ is simply-connected iff its boundary is connected [**15**]. With respect to (1) and (2) the sets ∂G_ν are in particular connected for $\nu \in \{1, \ldots, n\}$. So, condition (3) holds for example if $\bigcap_{\nu=1}^{n} \partial G_\nu \neq \emptyset$.

Proof of Theorem 1. For $m \in \mathbb{N}$ put $\mathcal{L}_m := \bigcup_{A \in \mathcal{F}_m} \partial A$. First we show by induction that $\mathcal{L}_m$ is connected for all $m \in \mathbb{N}$. Define $M := \bigcup_{\nu=1}^{n} \partial G_\nu$, then it follows from (4) that $\partial A \cap M \neq \emptyset$ for each $A \in \mathcal{F}_1$. Since every $A \in \mathcal{F}_1$ is simply-connected, ∂A is connected [**15**] and consequently

$$\mathcal{L}_1 = \bigcup_{A \in \mathcal{F}_1} \partial A = M \cup (\bigcup_{A \in \mathcal{F}_1} \partial A)$$

is connected, too. Now let $m \in \mathbb{N}$ and let $\mathcal{L}_m$ be connected. Due to the definition of $\mathcal{F}_m$, for each $A \in \mathcal{F}_{m+1} \setminus \mathcal{F}_m$ there is a $\mu \in \{1, \dots, n\}$ such that A is a component of $(R^{m+1})^{-1}(G_\mu)$. So, the set $R^m(A)$ is also a component of $R^{-1}(G_\mu)$ and because of (4) we can choose $p \in \partial R^m(A) \cap \partial G_{\nu_0}$ for some $\nu_0 \in \{1, \dots, n\}$. Because of the fact that A is a component of $(R^m)^{-1}(R^m(A))$, we have by Lemma 2:

$$\partial R^m(A) = R^m(\partial A)$$

That means, there is a $\tilde{p} \in \partial A$ such that $R^m(\tilde{p}) = p \in \partial G_{\nu_0}$. Hence by Lemma 2 we have

$$\partial A \cap \partial (R^m)^{-1}(G_{\nu_0}) = \partial A \cap (R^m)^{-1}(\partial G_{\nu_0}) \neq \emptyset.$$

Consequently, there is a component B of $(R^m)^{-1}(G_{\nu_0})$ such that $\partial A \cap \partial B \neq \emptyset$. Especially we get $\partial A \cap \mathcal{L}_m \neq \emptyset$. Because of (2), the set ∂A is connected and since $\mathcal{L}_m$ is connected too, this is also true for the set

$$\mathcal{L}_m \cup \Big(\bigcup_{A \in \mathcal{F}_{m+1} \setminus \mathcal{F}_m} \partial A \Big) = \bigcup_{A \in \mathcal{F}_{m+1}} \partial A = \mathcal{L}_{m+1}.$$

This finally shows, that $\mathcal{L}_m$ is connected for all $m \in \mathbb{N}$. Since $\mathcal{L}_m \subseteq \mathcal{L}_{m+1}$ for $m \in \mathbb{N}$, we get:

$$\bigcup_{A \in \mathcal{F}} \partial A = \bigcup_{m=1}^{\infty} \mathcal{L}_m \text{ is connected}$$

∎

As a corollary we obtain:

THEOREM 2. *Let $R: \mathbb{C}^* \to \mathbb{C}^*$ be a rational function, $deg(R) \geq 2$, let $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in \mathbb{C}^*$ pairwise distinct, attractive fixed-points of R and suppose*

(1) $\bigcup_{\nu=1}^{n} \partial A^*(\alpha_\nu)$ *is connected.*

(2) *Each component of* $\bigcup_{\nu=1}^{n} A(\alpha)$ *is simply-connected.*

(3) *For each component* A *of* $\bigcup_{\nu=1}^{n} R^{-1}(A^*(\alpha_\nu))$ *there is a* $\nu_0 \in \{1,\ldots,n\}$ *such that*

$$\partial A \cap \partial A^*(\alpha_{\nu_0}) \neq \emptyset.$$

Then $J(R)$ *is connected.*

Proof. For $m \in \mathbb{N}$ we put

$$\mathcal{F}_m := \bigcup_{\kappa=1}^{m} \{A \subseteq \mathbb{C}^* | A \text{ is a component of } (R^\kappa)^{-1}(A^*(\alpha_\nu)) \text{ for a } \nu \in \{1,\ldots,n\}\}$$

and $\mathcal{F} := \bigcup_{m=1}^{\infty} \mathcal{F}_m$. Obviously, $\mathcal{F}$ describes the set of all components of $\bigcup_{\nu=1}^{n} A(\alpha_\nu)$. Since $A^*(\alpha_\nu)$ is a component of $R^{-1}(A^*(\alpha_\nu))$ for $\nu \in 1,\ldots,n\}$, the conditions of Theorem 1 are fulfilled. So, we conclude the connectedness of $\bigcup_{A\in\mathcal{F}} \partial A$. It remains to prove that $J(R) = \overline{\bigcup_{A\in\mathcal{F}} \partial A}$, which gives the connectedness of $J(R)$.

For $x \in \bigcup_{A\in\mathcal{F}} \partial A$ there is a $m \in \mathbb{N}$ and a $\nu_0 \in \{1,\ldots,n\}$ such that $R^m(x) \in \partial A^*(\alpha_{\nu_0}) \subseteq J(R)$ holds.

Since $J(R) = R^{-1}(J(R))$ we get $x \in J(R)$ and consequently

$$\bigcup_{A\in\mathcal{F}} \partial A \subseteq J(R).$$

The closedness of $J(R)$ then implies $\overline{\bigcup_{A\in\mathcal{F}} \partial A} \subseteq J(R)$. Now, let $x \in J(R)$ and let U be a region with $x \in U$. Then we know from section 2 by Lemma 1:

$$\partial A^*(\alpha_1) \subseteq \bigcup_{\nu=1}^{\infty} R^\nu(U)$$

Especially, there is a $\nu_0 \in \mathbb{N}$ such that $R^{\nu_0}(U) \cap \partial A^*(\alpha_1) \neq \emptyset$. Since $R^{\nu_0}(U)$ is open, there exists z_0 with

$$z_0 \in U \cap (R^{\nu_0})^{-1}(A^*(\alpha_1)).$$

Choose $A \in \mathcal{F}_{\nu_0}$ with $z_0 \in A$. Then Lemma 2 implies

$$R^{\nu_0}(A) = A^*(\alpha_1).$$

Now we have $U \cap \partial A \neq \emptyset$, because it would follow

$$U = (U \cap A) \cup (U \cap (\mathbb{C}^* \setminus \overline{A}))$$

otherwise. But then the connectedness of U and $z_0 \in U \cap A$ yields $U \subseteq A$. So, we get $R^{\nu_0}(U) \subseteq R^{\nu_0}(A) = A^*(\alpha_1)$ in contradiction to $R^{\nu_0}(U) \cap \partial A^*(\alpha_1) \neq \emptyset$. Thus, for each region U with $x \in U$ there is an $A \in \mathcal{F}$ such that $U \cap \partial A \neq \emptyset$. So we have $x \in \overline{\bigcup_{A \in \mathcal{F}} \partial A}$ and consequently we obtain $J(R) \subseteq \overline{\bigcup_{A \in \mathcal{F}} \partial A}$. This completes the proof. ∎

Before applying the above theorem to Newton's method, a helpful criterion for later applications is given. It is a condition which guarantees that assumption (2) of Theorem 2 is fulfilled.

LEMMA 3. *Let $R: \mathbb{C}^* \to \mathbb{C}^*$ be a non-constant, rational function and let $\alpha \in \mathbb{C}^*$ be an attractive fixed-point of R. Furthermore, let C be the set of all critical points of R and suppose*

(1) *$A^*(\alpha)$ is simply-connected*

(2) *$C \cap A(\alpha) = C \cap A^*(\alpha)$.*

Then, all components of $A(\alpha)$ are simply-connected.

Proof. For $m \in \mathbb{N}$ we put now

$$\mathcal{F}_m := \bigcup_{\kappa=1}^{m} \{A \subseteq \mathbb{C}^* | A \text{ is a component of } (R^{\kappa})^{-1}(A^*(\alpha))\}$$

and $\mathcal{F} := \bigcup_{m=1}^{\infty} \mathcal{F}_m$. As noticed before, $\mathcal{F}$ is the set of all components of $A(\alpha)$. Then we have to show, that for each $m \in \mathbb{N}$ all $A \in \mathcal{F}_m$ are simply-connected. This is done by induction. Suppose $A \in \mathcal{F}_1$. If $A = A^*(\alpha)$, there is nothing left to be shown. So let $A \in \mathcal{F}_1 \setminus \{A^*(\alpha)\}$. Then according to (2) we have $A \cap C = \emptyset$, and $R_{|A}$ is locally injective. From Lemma 2 we get:

$$R_{|A}\colon A \to A^*(\alpha) \text{ is a homeomorphism}$$

Then, because of condition (1) the set A is simply-connected, too.

Let $m \in \mathbb{N}$ and suppose that each $A \in \mathcal{F}_m$ is simply-connected. Thus we consider $A \in \mathcal{F}_{m+1} \setminus \mathcal{F}_m$. So we have $A \neq A^*(\alpha)$ and again we get $A \cap C = \emptyset$. Since A is a component of $(R^{m+1})^{-1}(A^*(\alpha))$, the set $R(A)$ is a component of $(R^m)^{-1}(A^*(\alpha))$, which is simply-connected. So, it follows again from Lemma 2 that A and $R(A)$ are homeomorphic, and hence, A is simply-connected, too. ∎

4. An application to Newton's method for polynomials.

First of all, we want to combine known properties of Newton's method for polynomials. If $p\colon \mathbb{C} \to \mathbb{C}$ is a non-constant polynomial, let $N_p\colon \mathbb{C}^* \to \mathbb{C}^*$ the continuous extension of the mapping

$$z \mapsto z - \frac{p(z)}{p'(z)} \qquad (z \in \mathbb{C} \setminus (p')^{-1}(\{0\}))$$

to $\mathbb{C}^*$. The rational function N_p induced by the ploynomial p is called the **Newton-iteration-mapping** of p. From elementary calculus we have:

(1) $\mathrm{Fix}(N_p) = \{\infty\} \cup p^{-1}(\{0\})$

(2) If $\alpha \in \mathbb{C}$ is a zero of order $m \in \mathbb{N}$ of p, then

$$N_p'(\alpha) = 1 - \frac{1}{m}$$

(3) ∞ is a repelling fixed-point of N_p.

(4) $N_p^{-1}(\{\infty\}) \subseteq \{\infty\} \cup (p')^{-1}(\{0\})$

(5) The set C of all critical points of N_p is described by:

$$C = \{\alpha \in \mathbb{C} \mid p(\alpha) = 0 \wedge p'(\alpha) \neq 0\} \cup (p'')^{-1}(\{0\})$$

For an application of Theorem 2 we have to know under what condition the components of $A(\alpha)$ with $\alpha \in p^{-1}(\{0\})$, are simply-connected.

In case $C \cap A^*(\alpha) = \{\alpha\}$ and $N_p''(\alpha) \neq 0$, there exists a bijective, holomorphic function $\varphi\colon B(0,1) \to A^*(\alpha)$ such that

$$(*) \qquad (\varphi^{-1} \circ N_p \circ \varphi)(z) = z^2 \quad \text{for all } z \in B(0,1).$$

The latter result is an extension of a local statement of Boettcher [3], which can be established because of the fact that $C \cap A^*(\alpha) = \{\alpha\}$. The proof is quite the same as the proof of Theorem 9.5 in Blanchard [2] and uses the theory of ramified coverings.

Under the assumption $(*)$, both v. Haeseler [9] and Pommerenke [18] showed, that the radial limit $a := \lim_{\substack{r \to 1 \\ r \in (-1,1)}} \varphi(r)$ exists and that the point a is a repelling fixed-point of the iterated function. According to (1)–(3) this point has to be ∞ for N_p.

In case $\overline{O^+(C)} \cap \partial A^*(\alpha) = \emptyset$, v. Haeseler [9] was able to show that φ can be extended continuously to $\Phi\colon \overline{B(0,1)} \to \overline{A^*(\alpha)}$. The latter extension Φ is even a homeomorphism provided, that there exists a component A of $\mathbb{C}^* \setminus \overline{A^*(\alpha)}$ with $\infty \in \partial A$ and

$$\operatorname{card}(N_p^{-1}(\{w\}) \cap \overline{A^*(\alpha)}) \leq 2 \quad \text{for all } w \in \partial A^*(\alpha).$$

Using this, the dynamics of N_p on $\partial A^*(\alpha)$ can be described by the dynamics of $z \mapsto z^2$ on $\partial B(0,1)$ and because of

$$\partial B(0,1) = \overline{\bigcup_{n=1}^{\infty} \{z \in \partial B(0,1) | z^{2n} = 1\}}$$

we also have

$$\partial A^*(\alpha) = \overline{\bigcup_{n=1}^{\infty} \{z \in \partial A^*(\alpha) | N_p^n(z) = \infty\}}.$$

As a resumé we have:

LEMMA 4. *Let $p: \mathbb{C} \to \mathbb{C}$ be a polynomial and let C be the set of all critical points of N_p, $\alpha \in p^{-1}(\{0\})$ such that*

$$a)\ C \cap A^*(\alpha) = \{\alpha\}$$

$$b)\ N_p''(\alpha) \neq 0.$$

Then there is a bijective, holomorphic function φ: $B(0,1) \to A^*(\alpha)$ with

$$i)\ (\varphi^{-1} \circ N_p \circ \varphi)(z) = z^2 \text{ for all } z \in B(0,1)$$

$$ii)\ \lim_{\substack{r \to 1 \\ r \in (-1,1)}} \varphi(r) = \infty.$$

If there is a component A of $\mathbb{C}^* \setminus \overline{A^*(\alpha)}$ with $\infty \in \partial A$ and if $\overline{O^+(C)} \cap \partial A^*(\alpha) = \emptyset$ as well as

$$\text{card}(N_p^{-1}(\{w\}) \cap \overline{A^*(\alpha)}) \leq 2 \quad \text{for all } w \in \partial A^*(\alpha),$$

then there is a homeomorphism Φ: $\overline{B(0,1)} \to \overline{A^*(\alpha)}$ with $\Phi_{|B(0,1)} = \varphi$ and we have

$$\partial A^*(\alpha) = \overline{\bigcup_{n=1}^{\infty} \{z \in \partial A^*(\alpha) | N_p^n(z) = \infty\}}.$$

Now, all preparations for the application of Theorem 2 to Newton's method for polynomials of degree 3 are made and we get:

THEOREM 3. *Let $p: \mathbb{C} \to \mathbb{C}$ be a cubic polynomial and let C be the set of all critical points of N_p. Suppose*

$$(+) \qquad \overline{O^+(C)} \cap \bigcup_{\alpha \in p^{-1}(\{0\})} \partial A^*(\alpha) = \emptyset.$$

Then $J(N_p)$ is connected.

Proof. We distinguish three cases:

Case a): Suppose that p has the pairwise distinct zeros $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Then we have for $j \in \{1, 2, 3\}$:

$$p'(\alpha_j) \neq 0 \text{ and so } N_p'(\alpha_j) = 0$$

Since $\deg(p'') = 1$, there is one and only one $\beta \in (p'')^{-1}(\{0\})$ and it follows that:

$$C = \{\alpha_1, \alpha_2, \alpha_3, \beta\}$$

Thus there are two zeros of p, without loss of generality α_1 and α_2, satisfying:

$$(1) \quad C \cap A(\alpha_1) = \{\alpha_1\} \ \wedge \ C \cap A(\alpha_2) = \{\alpha_2\}$$

Especially, we may assume $p''(\alpha_j) \neq 0$ for $j \in \{1, 2\}$ and therefore $N_p''(\alpha_j) \neq 0$ for $j \in \{1, 2\}$. Lemma 4 yields:

$$A^*(\alpha_1) \text{ and } A^*(\alpha_2) \text{ are simply-connected}$$
$$\text{and } \infty \in \partial A^*(\alpha_1) \cap \partial A^*(\alpha_2)$$

Consequently, we have by (1) and Lemma 3:

Each component of $A(\alpha_1) \cup A(\alpha_2)$ is simply-connected, and since $\partial A^*(\alpha_1) \cap \partial A^*(\alpha_2) \neq \emptyset$ the set $\partial A^*(\alpha_1) \cup \partial A^*(\alpha_2)$ is connected. The rational function N_p takes the value ∞ exactly three times (counting multiplicities).

Consequently, both α_1 and α_2 are taken exactly three times (c.m.). Since for $j \in \{1,2\}$ the map N_p takes the value α_j excatly twice in α_j, there is precisely one $\tilde{\alpha}_j \in \mathbb{C} \setminus \{\alpha_j\}$ such that $N_p(\tilde{\alpha}_j) = \alpha_j$.

According to Lemma 4, it follows for $z \in A^*(\alpha_j)$ with $N_p(z) = \alpha_j$ that $z = \alpha_j$. So, we conclude that $\tilde{\alpha}_j \notin A^*(\alpha_j)$ and besides $A^*(\alpha_j)$ there exists precisely one component A_j of $N_p^{-1}(A^*(\alpha_j))$ with $A_j \neq A^*(\alpha_j)$.

In case $\partial A_{j_0} \cap \partial A^*(\alpha_{j_0}) \neq \emptyset$ for some $j_0 \in \{1,2\}$ the proposition follows immediately from Theorem 2. So from now on we assume for $j \in \{1,2\}$:

$$(2) \qquad \partial A_j \cap \partial A^*(\alpha_j) = \emptyset$$

Since each value of $\mathbb{C}^*$ is taken by N_p exactly three times (c.m.), it follows from (2) and Lemma 2 for $j \in \{1,2\}$ and $w \in \partial A^*(\alpha_j)$:

$$\operatorname{card}(N_p^{-1}(\{w\}) \cap \overline{A^*(\alpha_j)}) \leq 2$$

So all of the conditions of Lemma 4 are fulfilled and we obtain:

$$(3) \qquad \partial A^*(\alpha_2) = \overline{\bigcup_{n=1}^{\infty} \{z \in \partial A^*(\alpha_2) | N_p^n(z) = \infty\}}$$

Now, suppose that the following inclusion holds:

$$(*) \qquad \bigcup_{n=1}^{\infty} \{z \in \partial A^*(\alpha_2) | N_p^n(z) = \infty\} \subseteq \partial A^*(\alpha_1)$$

Then from (3) it follows:

$$\partial A^*(\alpha_2) \subseteq \overline{\partial A^*(\alpha_1)} = \partial A^*(\alpha_1)$$

Lemma 4 shows, that $\partial A^*(\alpha_1)$ is the homeomorphic image of the unit circle S^1. Hence, it follows from Jordans' theorem that $\mathbb{C}^* \setminus \overline{A^*(\alpha_1)}$ is a simply-connected region. As a result of $A^*(\alpha_2) \subseteq \mathbb{C}^* \setminus \overline{A^*(\alpha_1)}$ and $\partial A^*(\alpha_2) \subseteq \partial A^*(\alpha_1)$ we immediately get

$$A^*(\alpha_2) = \mathbb{C}^* \setminus \overline{A^*(\alpha_1)}.$$

But the latter is a contradiction to $A^*(\alpha_3) \neq \emptyset$. Consequently assumption $(*)$ is false and hence, there is an $n \in \mathbb{N}$, such that

$$(4) \qquad \{z \in \partial A^*(\alpha_2) | N_p^n(z) = \infty\} \setminus \partial A^*(\alpha_1) \neq \emptyset \text{ holds.}$$

It should be noticed, that the latter argumentation needs essentially the connectedness of $\mathbb{C}^* \setminus \overline{A^*(\alpha_1)}$ which follows in particular from Jordan's theorem. In general there are simply-connected regions $G_1, G_2 \subseteq \mathbb{C}^*$ with $G_1 \cap G_2 = \emptyset$, $\partial G_1 = \partial G_2$ and $G_1 \neq \mathbb{C}^* \setminus \overline{G_2}$, c.f. **[12]**, **[14]**.

Now let $n_0 \in \mathbb{N}$ be minimal having property (4). Then there is a point $z_0 \in \partial A^*(\alpha_2) \setminus \partial A^*(\alpha_1)$ with $N_p^{n_0}(z_0) = \infty$ and $N_p(z_0) \in \partial A^*(\alpha_1) \cap \partial A^*(\alpha_2)$. Let A be the component of $N_p^{-1}(A^*(\alpha_1))$ with $z_0 \in \partial A$. Since $z_0 \notin \partial A^*(\alpha_1)$, we get $A \neq A^*(\alpha_1)$ and thus $A = A_1$. Because of the fact that $z_0 \in \partial A^*(\alpha_2)$ we now conclude

$$\partial A_1 \cap \partial A^*(\alpha_2) \neq \emptyset.$$

In the same way we get $\partial A_2 \cap \partial A^*(\alpha_1) \neq \emptyset$, such that all conditions of Theorem 2 are established. Hence, $J(N_p)$ is connected.

<u>Case b):</u> Suppose that p has a simple root $\alpha_1 \in \mathbb{C}$, as well as a zero of multiplicity two, say $\alpha_2 \in \mathbb{C}$. Then $N_p'(\alpha_1) = 0$, but $N_p'(\alpha_2) = \frac{1}{2}$. Once again, there is one and only one $\beta \in (p'')^{-1}(\{0\})$, such that we have: $C = \{\alpha_1, \beta\}$

Since α_2 is an attractive fixed-point we necessarily obtain $\beta \in A^*(\alpha_2)$ from Lemma 1,(6). Hence we get: $C \cap A^*(\alpha_1) = \{\alpha_1\}$

Again, we have $p''(\alpha_1) \neq 0$ and consequently $N_p''(\alpha_1) \neq 0$. Lemma 4 now shows that $A^*(\alpha_1)$ must be simply-connected. Now, N_p takes the value ∞ exactly twice (c.m.), hence the value α_1 is taken exactly twice, as well. Since this is already true at the point α_1, the set $A^*(\alpha_1)$ is the only component of $A(\alpha_1)$ that is $A(\alpha_1) = A^*(\alpha_1)$. Because $A^*(\alpha_1)$ is simply-connected, we obtain that $J(N_p) = \partial A(\alpha_1) = \partial A^*(\alpha_1)$ is connected.

Case c): Suppose that p has a zero $\alpha \in \mathbb{C}$ of multiplicity three. Then we have :

$$N_p(z) = \alpha + \frac{2}{3}(z - \alpha) \qquad \text{for all } z \in \mathbb{C}$$

So we get $A(\alpha) = \mathbb{C}$ and $J(N_p) = \{\infty\}$. This completes the proof. ∎

Remarks.

a) The condition (+) seems to be fulfilled for almost all cubic polynomials. This is suggested by the computer study of Curry, Garnett and Sullivan [**6**].

b) From the connectedness of $J(N_p)$ we especially conclude, that all components of $F(N_p)$ are simply-connected. In particular, there are no so called Herman-rings (c.f. Blanchard [2]) and each immediate basin of attraction has to be simply-connected. Moreover, if α is an attractive period cycle (i.e. there is a $k \in \mathbb{N}$ such that α is an attractive fixed-point of N_p^k), the immediate basin of attraction with respect to N_p^k is a component of $F(N_p)$. So, it has to be simply-connected and with the aid of Riemann's mapping theorem the dynamics of N_p^k on this immediate basin is conjugated to the dynamics of a Blaschke product of finite order [**9**].

Acknowledgement.

I want to thank Prof. Dr. H. Th. Jongen for the numerous encouraging and stimulating conversations without which this paper would not have been written.

REFERENCES

[1] H.E. BENZINGER, S.A. BURNS AND J.I. PALMORE, *Chaotic complex dynamics and newton's method*, Phys. Lett. A **119**, 9 (1987), pp. 441–446.

[2] P. BLANCHARD, *Complex analytic dynamics on the Riemann sphere*, Bull. Amer. Math. Soc. **11** (1984), pp. 85–141.

[3] F. E. BOETTCHER, *The principal laws of convergence and their application to analysis*, Bull. Kasan Math. Soc. **14** (1905).

[4] H. BROLIN, *Invariant sets under iteration of rational functions*, Arkiv Math. **6** (1965), pp. 103–144.

[5] A. CAYLEY, *The Newton-Fourier imaginary problem*, Amer. Jour. Math. II **97** (1879).

[6] J.H. CURRY, L. GARNETT AND D.R. SULLIVAN, *On the iteration of a rational function: computer experiments with Newton's method*, Comm. Math. Phys. **91** (1983), pp. 267–277.

[7] P. FATOU, *Sur les solution uniformes de certaines équations fonctionelles*, C.R. Acad. Sci. Paris **143** (1906), pp. 546–548.

__________, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **47** (1919), pp. 161–271.

__________, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **48** (1920), pp.33–94.

__________, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **48** (1920), pp.208–314.

[8] O. FORSTER, *Riemannsche Flächen*, Springer Berlin (1977).

[9] F. v. HAESELER, *Über sofortige Attraktionsgebiete superattraktiver Zyklen*, Dissertation Universität Bremen (1985).

[10] M.R. HERMAN, *Are there critical points on the boundaries of singular domains?*, Comm. Math. Phys. 99 (1985), pp. 593-612.

[11] G. JULIA, *Mémoire sur l'iteration des fonctions rationelles*, J. Math. Pure Appl. 8 (1918), pp. 47–245.

[12] B. KNASTER, *Quelques coupres singulières du plan*, Fund. Math. 7 (1925), pp. 264–289.

[13] M.G. KOENIGS, *Recherches sur les integrals de certains equations fonctionnelles*, Ann. Sci. Ecole Norm Sup. 1 3 (1884), pp. 1–41.

[14] KURATOWSKI, *Topology II*, Academic press New York and London (1968), p. 560.

________________, *Sur les coupures irréductibles du plan*, Fund. Math. 6 (1924), p. 138.

________________, *Sur la structure des frontières communes à deux régions*, Fund. Math. 12 (1928), p. 36.

[15] M. NEWMAN, *Elements of the topology of plane sets of points*, Cambridge university press (1964).

[16] H.O.PEITGEN, D. SAUPE AND F. V. HAESELER, *Cayley's Problem and Julia Sets*, Math. Intell. 6 (1984), pp. 11–20.

[17] H.O. PEITGEN AND P. RICHTER, *The beauty of fractals*, Springer Berlin (1986).

[18] CH. POMMERENKE, *On conformal mapping and iteration of rational functions*, Complex Variables 4 (1986), pp. 117–126.

RWTH Aachen Lehrstuhl C für Mathematik Templergraben 55 D-5100 Aachen